Multiple phase transitions in long range first-passage percolation on square lattice

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Joint with Partha Dey

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Consider $\mathbb{Z}^d$ with nearest neighbor edges, where each edge has an i.i.d. nonnegative weight from a fixed distribution $F$. 
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For a path $\mathcal{P}$, the passage time for $\mathcal{P}$ is defined as the sum of weights over all the edges in $\mathcal{P}$.

For $x, y \in \mathbb{Z}^d$, the first-passage time $a(x, y)$ is defined as the minimum passage time over all paths from $x$ to $y$. 
The model was introduced by Hammersley and Welsh in 1965, where they proved that for all $x \in \mathbb{Z}^d$

$$\nu(x) = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}(a(0, nx))$$

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The shape theorem by Cox and Durrett ('81) says that

$$\frac{1}{t} \{ x \in \mathbb{Z}^d : a(0, x) \leq t \} \oplus \left[ -\frac{1}{2}, \frac{1}{2} \right]^d \xrightarrow{t \to \infty} B,$$

where $B = \{ x : \nu(x) \leq 1 \}$ is a convex subset of $\mathbb{R}^d$. 
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S. Chatterjee  
LRFPP
Mean behavior

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\frac{1}{t} \{ x \in \mathbb{Z}^d : a(0, x) \leq t \} \oplus \left[ -\frac{1}{2}, \frac{1}{2} \right]^d \overset{t \to \infty}{\longrightarrow} B,
\]

where \( B = \{ x : \nu(x) \leq 1 \} \) is a convex subset of \( \mathbb{R}^d \).

\( \text{Var}(a(0, nx)) = o(n) \).
Figure: Limiting shape for \( \{ x \in \mathbb{Z}^d : a(0, x) \leq t \} \).
Long-range First-Passage Percolation (LRFPP) on $\mathbb{Z}^d$

- Let $|| \cdot ||$ be the $\ell_1$ norm on $\mathbb{Z}^d$. 
Let $\| \cdot \|$ be the $\ell_1$ norm on $\mathbb{Z}^d$.

Now consider the complete graph on $\mathbb{Z}^d$ with unoriented edge set $E = \{ \langle xy \rangle : x, y \in \mathbb{Z}^d, x \neq y \}$.

Let $\{ W_e : e \in E \}$ be a collection of i.i.d. mean one exponentially distributed random variables.

For a self avoiding path $p = \langle x_0 x_1 \ldots x_k \rangle$ with $k$ edges, define its $\alpha$-th passage time $W_p^\alpha := \sum_{i=1}^{k} \| x_i - x_{i-1} \|^\alpha W_{\langle x_{i-1} x_i \rangle}$.

For $x, y \in \mathbb{Z}^d$, the $\alpha$-th first-passage time is

$$T^\alpha(x, y) := \inf_{p \in \mathcal{P}_{xy}} W_p^\alpha,$$

where $\mathcal{P}_{xy}$ is the set of all self-avoiding paths joining $x$ and $y$.

**Question:** How does $T^\alpha(0, x)$ behave as $\| x \|$ grows?
Alternative formulation for $\alpha > d$

$\alpha > d$ implies $\sum_{x \in \mathbb{Z}^d} ||x||^{-\alpha} < \infty$.

- This formulation is an extension of Richardson’s model.
- Each site of $\mathbb{Z}^d$ is either occupied or vacant.
- Initially the origin is occupied only.
- Once $x$ is occupied, it attempts to communicate at rate 1 and in each attempt it chooses a site $y$ with probability $c||x - y||^{-\alpha}$ and makes it occupied.
- Occupied sites stay occupied.

**Question:** If $B^\alpha_t$ is the set of vertices occupied by time $t$, then how does it grow?
Growth dynamics cartoon
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Mollison (1972) has considered similar models in the context of spatial propagation of epidemics. He proves linear growth in $d = 1$ for $\alpha > 3$.

Cannas, Marco and Montemurro (2006) have considered long distance dispersal models in the context of biological invasion.

Aldous (2007) has considered similar models on a torus in the context of random percolation of information through agent networks, and studied various related game theoretic aspects.

C. and Durrett (2011) have considered a related continuous model (short-long FPP) on torus.

Barbour and Reinert (arXiv) have considered gossip models on smooth Riemannian manifold.
Long-range percolation (LRP) results

\( x, y \in \mathbb{Z}^d \) are connected by an edge independently with prob.
\[
p_{xy} = |x - y|^{-\alpha + o(1)} \quad \text{as} \quad |x - y| \to \infty.
\]
Let \( D(., .) \) be the associated random metric, \( B(x, k) \) be the balls
and \( D_L \) be the diameter of connected component of \([-L, L]^d\).

- **LRP in one dimension:**
  - Schulman (’83):
  - Aizenman and Newman, Newman and Schulman (’86)
  - Imbrie and Newman (’88)
  - Benjamini and Berger (’01):

- Coppersmith, Gamarnik and Sviridenko (’02): Diameter of LRP clusters.

- Biskup (2004, arXiv): Behavior of graph distance and \( D_L \) for \( d < \alpha < 2d \).

- Trapman (2010): limits of \(|B(0, k)|^{1/k}\) as \( k \to \infty \).

- Berger (arXiv): Lower bound for \( D(., .) \) when \( \alpha > 2d \)
Long-range percolation (LRP) results

$x, y \in \mathbb{Z}^d$ are connected by an edge independently with prob.

\[ p_{xy} = |x - y|^{-\alpha + o(1)} \text{ as } |x - y| \to \infty. \]

Let $D_L$ be the diameter of connected component of $[-L, L]^d$ in the associated random metric.

\[
D_L \begin{cases} 
\to \left[ \frac{\alpha}{(d - \alpha)} \right], & \alpha < d \\
\asymp \log L / \log \log L, & \alpha = d \\
= (\log L)^{\Delta(\alpha) + o(1)}, & d < \alpha < 2d \\
= L^{\theta(\beta) + o(1)} (expected!), & d < \alpha < 2d \\
\asymp L (expected!), & \alpha > 2d.
\end{cases}
\]

(Cor. of Benjamini, Kesten Peres and Schramm (’04))

(Coppersmith, Gamarnik and Sviridenko (’02))

(Biskup (arXiv))

for $p_{xy} = \beta |x - y|^{-2d}$ for $\alpha > 2d$. 

S. Chatterjee LRFPP
Simulation for $\mathcal{B}_t^\alpha = \{x \in \mathbb{Z}^2 : T^\alpha(0, x) \leq t\}$

Figure: Growth for $\alpha = 3$ (top two), 3.5 (bottom two)
Simulation for $B^\alpha_t = \{x \in \mathbb{Z}^2 : D^\alpha(0, x) \leq t\}$

Figure: Growth for $\alpha = 4$ (top two), 4.5 (bottom two)
Simulation for $B_t^\alpha = \{ x \in \mathbb{Z}^2 : D^\alpha(0, x) \leq t \}$

**Figure:** Growth for $\alpha = 5$ (top two), 5.5 (bottom left), 6 (bottom right)
Conclusions of Cannas, Marco and Montemurro (2006)

As can be seen in the vegetation cover map of *C. grandiflora* in Fig. 9, the overall spatial pattern predicted by the simulation (see Fig. 1) appears in a very large spatial scale (hundreds of km, Fig. 9), i.e., a main patch with an irregular border, centered around the focus of the introduction (Charter Towers [21]). Surrounded by a distribution of secondary patches. A similar pattern, but in a much smaller spatial scale can be observed in a vegetation cover map for *P. ponderosa* (several km, Fig. 2 in Ref. [22]). This is consistent with the time scale associated with the spread of each species (more than 100 years for *C. grandiflora* [21] and around 30 years for *P. ponderosa* [22]).

In order to obtain a quantitative comparison we first calculate the fractal dimension of the main patch border in both cases. To this end, we digitized both images and isolated the border of all patches. Then we applied the box counting algorithm to the borders, i.e., we calculated the number of boxes needed to cover only the patch border. The fact that all patches present a similar

**Figure**: Abundance of *C. grandiflora* and box counting plot

**Conclusions**: Based on the existence of second moment of the dispersal distribution, “$\alpha > 2d$” behavior of the model is same as that of the short range models.

For two dimension, the box counting dimension for the boundary curve is independent of $\alpha$ for $2 < \alpha < 3$ and decreasing for $3 < \alpha < 4$. 
Our result: Phase transition

- **Instantaneous growth regime:** $\alpha < d$.
  Here $B_t^\alpha = \mathbb{Z}^d$ for any $t > 0$ with probability 1.

- **Stretched exponential growth regime:** $d < \alpha < 2d$.
  Here, the diameter of $B_t^\alpha$ is $\exp(t^{1/\Delta + o(1)})$, where $\Delta = \log 2 / \log(2d/\alpha)$ which increases from 1 to $\infty$ as $\alpha$ goes from $d$ to $2d$.

- **Superlinear growth regime:** $2d < \alpha < 2d + 1$.
  Here, the diameter of $B_t^\alpha$ is $t^{1/(\alpha-2d)+o(1)}$, so the index decreases from $\infty$ to 1 as $\alpha$ goes from $2d$ to $2d + 1$.

- **Linear Growth regime:** $\alpha > 2d + 1$.
  Here, the diameter of $B_t^\alpha$ is $t^{1+o(1)}$.

**Remarks:**

“Superlinear regime” disproves the first conclusions of CMM (’06). Phase transitions in the LRP and LRFPP models are not identical.
Instantaneous growth, $\alpha < d$

We show $P(T^\alpha(0, x) \leq t) = 1$ for any $t > 0$ and $x \in \mathbb{Z}^d$.

- Fix an integer $K > d/(d - \alpha)$ and let $\ell_j = 2^j(k - 1)^j||x||$.
- Let $B_i^{(j)} := \{y \in \mathbb{Z}^d : (2i - 1)\ell_j \leq ||y|| \leq 2i\ell_j\}$ and

\[ P_j := \{\pi = \langle 0x_1 \ldots x_{K-1}x \rangle : x_i \in B_i^{(j)}\}. \]
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  \[P_j := \{\pi = \langle 0x_1 \ldots x_{K-1}x \rangle : x_i \in B_i^{(j)}\}\] 

- Using second moment argument 
  \[P \left( \inf_{\pi \in P_j} W^\alpha_{\pi} \leq t \right) \geq \left( \frac{EN_j}{EN_j^2} \right)^2, \text{ where } N_j := |P_j \cap \{\pi : W^\alpha_{\pi} \leq t\}|.\]

- Using tail bounds for sum of independent exponential random variables we can lower bound $E(N_j)$ and upper bound $E(N_j^2)$ 
  \[P \left( \inf_{\pi \in P_j} W^\alpha_{\pi} \leq t \right) \geq \delta > 0 \text{ independent of } j.\]

- So $P(T^\alpha(0, x) > t) \leq \prod_j P(\inf_{\pi \in P_j} W^\alpha_{\pi} > t) = 0$. 
For $\alpha \in (d, 2d)$, recall $\Delta := 1/ \log_2(2d/\alpha) \in (1, \infty)$. We show $P(T^\alpha(0, x) \leq t) \leq \exp \left( c_1 t^{1/\Delta+\varepsilon} - c_2 \ln \|x\| \right)$ for small $\varepsilon > 0$, so for $\alpha \in (d, 2d)$, $T^\alpha(0, x) \geq (\ln \|x\|)^{\Delta-\varepsilon}$ w.h.p.
Upper bound for $\text{Diam}(B_t^\alpha)$

For $\alpha \in (d, 2d)$, recall $\Delta := 1/ \log_2(2d/\alpha) \in (1, \infty)$. We show $P(T^\alpha(0, x) \leq t) \leq \exp \left( c_1 t^{1/\Delta + \varepsilon} - c_2 \ln ||x|| \right)$ for small $\varepsilon > 0$, so for $\alpha \in (d, 2d)$, $T^\alpha(0, x) \geq (\ln ||x||)^{\Delta - \varepsilon}$ w.h.p.

**Sketch:** Let $g(u) := E|B_u^\alpha|$ be the expected volume. Let $N(x)$ be the number of edges in the optimal path from $0$ to $x$. $N(x)$ small means a long edges has been used in the optimal path:

$$P(T^\alpha(0, x) \leq t, N(x) \leq a \cdot t) \leq ||x||^{-\alpha}(at)^\alpha \int_0^t g(y)[g(t-y)-1] \, dy.$$ 

$N(x)$ large means too many edges are used within small time. Using large deviation estimates:

$$P(T^\alpha(0, x) \leq t, N(x) > a \cdot t) \leq c||x||^{-\alpha} \exp(-\delta(a)t).$$

Combining (1) and (2) and using $g(t) = \sum_x P(T^\alpha(0, x) \leq t)$ get a recursive inequality for $g(\cdot)$. Solve it.
**Sketch:** Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) with \( 2f(x) < x \).

Fix \( x \) with \( ||x|| = n \). Define \( f_0 = n \) and \( f_k = f(f_{k-1}) \) inductively.

Min weight of an edge between \( B(0, f_1) \) and \( B(x, f_1) \) is exponential with rate

\[
\sum_{u \in B(0, f_1), v \in B(x, f_1)} |v - u|^{-\alpha} \sim (n - 2f_1)^{-\alpha}f_1^{2d}.
\]

Let the end points for the minimal edge be \( u_1, v_1 \) respectively.

Consider minimal edge between \( B(0, f_2), B(u_1, f_2) \) and between \( B(v_1, f_2), B(X, f_2) \) and proceed similarly.

After \( k \) steps, we have \( 2^k \) balls of radius \( f_k \) and the optimal time is upper bounded by

\[
\sum_{i=1}^{k} 2^{i-1}(f_i-1 - 2f_i)^{\alpha}f_i^{-2d} + 2^k f_k.
\]

We have to optimize this over the function \( f \) and \( k \).
For $d < \alpha < 2d$, we take $f(x) = x^\gamma$ and the optimal \( \gamma = \alpha/2d \).

The optimal $k$ is such that $f_k \approx 1$.

Here the upper bound for $T^\alpha(0, x)$ shows that
\[
P(T^\alpha(0, x) \geq (\log ||x||)^{\Delta + \varepsilon}) \to 0 \text{ fast for any } \varepsilon > 0.
\]

For $2d < \alpha < 2d + 1$, we take $f(x) = x/a$ with $a > 2$ and the optimal $a$ goes to 2 as $\alpha \uparrow 2d + 1$.

The optimal $k$ is again such that $f_k \approx 1$.

The upper bound on $T^\alpha(0, x)$ gives
\[
P(T^\alpha(0, x) \geq ||x||^{\alpha-2d+\varepsilon}) \to 0 \text{ fast for any } \varepsilon > 0.
\]
Upper bound for diameter for $2d < \alpha < 2d + 1$

Our previous technique of estimating $P(T^\alpha(0, x) \leq t)$ gives

$$P(T^\alpha(0, x) \leq t) \leq C(t^\gamma/\|x\|)^\alpha$$

for some $\gamma > 1/(\alpha - 2d)$, which does not give matching upper bound for diameter.

We show that if $P(Diam(B^\alpha_t) \leq t^\gamma) \to 1$ for some $\gamma > 1/(\alpha - 2d)$, then we can improve $\gamma$ recursively to have eventually

$$P(Diam(B^\alpha_t) \leq t^{1/(\alpha-2d)+\varepsilon}) \to 1$$

for any $\varepsilon > 0$. 
Suppose $P(\text{Diam}(\mathcal{B}_t^\alpha) \leq t^\gamma) \rightarrow 1$ for some $\gamma > 1/(\alpha - 2d)$.

W.h.p. no edge of length $\geq t^\delta$ will be used till time $t$, where $(\gamma d + 1)/(\alpha - d) < \delta$.

On the above ‘good’ event, $x \in \mathcal{B}_t^\alpha$ implies

$$t \geq T^\alpha(0, x) \geq \inf_{p \in P_{0,x}: \text{no edge } \geq t^\delta} \sum_{e \in p} |e|^\alpha W_e \geq t^{-\delta(\beta - \alpha)} T^\beta(0, x).$$

So $\mathcal{B}_t^\alpha \subset \mathcal{B}_t^{\beta_{1+\delta(\beta - \alpha)}}$ w.h.p.

If $\beta$ just crosses $2d + 1$, then assuming linear growth of $T^\beta(0, x)$ w.h.p. the diameter of $\mathcal{B}_t^\alpha$ is at most $1 + \delta(\beta - \alpha) < \gamma$. 
Linear growth for $\alpha > 2d + 1$

- Nearest neighbor growth ensures at most linear growth for $T^{\alpha}(0, x)$.
- In view of subadditivity, it suffices to show at least linear growth for $T^{\alpha}(0, x)$.
- The key is that if 
  \[ \theta \in \left(\frac{d + 1}{\alpha - d}, 1\right), \]
then no edge of length $n^\theta$ is used within time $n$ by a vertex in the Euclidean ball of diameter $n$.
- Divide $[0, n]^d$ into a grid of size $n^\theta$, and the optimal path from $0$ to $ne_1$ must jump from a box to a nearest neighbor box.
- It crosses $O(n^{1-\theta})$ such boxes and takes at least $O(n^\theta)$ time for a fraction of them by a renormalization argument.
Future questions

- Study the critical values $\alpha = d, 2d, 2d + 1$.
- Properties of rescaled growth set.
- Bounds for boundary fluctuations.
- On a torus study the time evolution of the fraction covered.
Thank You