Asymptotic Behavior of Aldous’ Gossip Process

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First passage percolation on a torus

- Space is $\Lambda(N) = (\mathbb{Z} \mod N)^2$.
- Suppose one agent is present at each vertex of a $\Lambda(N)$.
- At time 0 the center receives an information.
- Each neighbor of the center gets the information independently at rate $1/4$.
- In general, whenever a vertex is informed, each of its uninformed neighbor gets the information independently at rate $1/4$.
- $\xi_t$ is the set of vertices informed by time $t$. $\xi_0 = \{(0,0)\}$. 

Questions:

How does $\xi_t$ grow?

When $\xi_t = \Lambda(N)$?
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Results for nearest neighbor case

- (HW(’65) and Kesten(’86)) In any direction the information percolates with a linear speed.
- (Cox and Durrett(’81)) Diameter of $\xi_t$ grows linearly and it has an asymptotic shape.
- Fluctuation results for the minimum passage time (on lattices):
  - lower bound due to Pemantle and Peres(’94), Newman and Piza(’95) and Zhang(’08).
  - upper bound due to of Benjamini, Kalai and Schramm(’03).
  - conjectured behavior differs from both bounds.

If $T_N$ is the time when every agent on the torus is informed, then $T_N/N$ converges to a number.
State of the process is $\xi_t \subset \Lambda(N)$, the set of informed vertices at time $t$. $\xi_0 = \{(0, 0)\}$.

Information spreads from vertex $i$ to $j$ at rate $\nu_{ij}$, where

$$\nu_{ij} = \begin{cases} 1/4 & \text{if } j \text{ is a (nearest) neighbor of } i \\ \lambda_N/(N^2 - 5) & \text{if not.} \end{cases}$$

If a vertex gets the information from a non-neighbor, we call it a ‘center’.

So each informed vertex tries to give birth to new centers at rate $\lambda_N$. 
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**Question:** How does $\xi_t$ grow? How does $T_N$ scale?
Our model (‘balloon process $C_t$’)

To simplify:

- we remove randomness from nearest neighbor growth
- we formulate on the (real) torus $\Gamma(N) = (\mathbb{R} \mod N)^2$.

The state of our process at time $t$ is $C_t \subset \Gamma(N)$, the subset informed by time $t$.

$C_t$ starts with one center chosen uniformly from $\Gamma(N)$ at time 0.

Each center corresponds to a disk, whose radius grows as $r(s) = s/\sqrt{2\pi}$.

At time $t$, birth rate of new centers is $\lambda N |C_t| = \lambda NC_t$.

The location of each new center is chosen uniformly from the torus. If the new center lands at $x \in C_t$, it has no effect. But we count all centers in $\tilde{X}_t$. 

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Aldous’ Gossip Process  

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- If the new center lands at $x \in C_t$, it has no effect. But we count all centers in $\tilde{X}_t$.
Phase transition

Consider $\lambda_N = N^{-\alpha}$.

- **Case 1: $\alpha > 3$.**
  - If the diameter of $C_t$ grows linearly, then $\int_0^N C_t \, dt = O(N^3)$.
  - So with high probability there is no long jump before the initial disk covers the entire torus, and
  - the cover time satisfies
    \[
    \frac{T_N}{N} \xrightarrow{P} \sqrt{\pi}.
    \]
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- **Case 2**: \( \alpha = 3 \).
  - with probabilities bounded away from 0, (i) no long range jump and \( T_N \approx \sqrt{\pi}N \), and (ii) there is one that lands close enough to \((N/2, N/2)\) to make \( T_N \leq (1 - \delta)\sqrt{\pi} N \).
  - \( T_N/N \) converges weekly to a random variable with support \([0, \sqrt{\pi}]\) and an atom at \( \sqrt{\pi} \).

- **Case 3**: \( \alpha < 3 \).
  - Many long range jumps.
  - The cover time is significantly accelerated.
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Branching balloon process $A_t$

- Overlaps among the disks in $C_t$ make it difficult to study.
- We begin by studying much simpler balloon branching process $A_t$.

In the process $A_t$,

- we do not ignore any center (unlike in $C_t$),
- $A_t$ and $X_t$ denote the sum of the areas of all of the disks and the number of centers at time $t$,
- new centers are born at rate $N^{-\alpha}A_t$ at uniformly chosen locations.

We couple $C_t$ and $A_t$ so that

- they start from the same point, and
- $C_t \subset A_t$, $C_t \leq A_t$, $\tilde{X}_t \leq X_t \ \forall t \geq 0$. (Recall $\tilde{X}_t = \#$ centers in $C_t$)
Properties of $A_t$

Let $\lambda = N^{-\alpha}$.

- Let $L_t := \int_0^t X_s \, ds$ be the length process. Then
  $$A_t = \int_0^t (t - s)^2 / 2 \, dX_s = \int_0^t L_s \, ds.$$  
- Using i.i.d. behavior of all the centers,
  $$X_t = 1 + \sum_{i : s_i \in \Pi_t} X_{t-s_i}^i,$$
  where $\Pi_t \subset [0, t]$ is the set of time points when the initial disk gives birth to new centers, and $X_i$'s are i.i.d. copies of $X$.
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- A little Poisson process computation shows that
  
  $$EX_t = 1 + \int_0^t EX_{t-s} \frac{\lambda s^2}{2} \, ds,$$

  as the initial disk has area $s^2/2$ at time $s$.

- Solving the renewal equation
  
  $$EX_t = \sum_{k=0}^{\infty} \frac{\lambda^k t^{3k}}{(3k)!}.$$
Properties of $A_t$ (continued)

- Solving the ODE $v''' = \lambda v$, ($\omega, \omega^2$ are complex cube roots of 1)
  \[ EX_t = \frac{1}{3} \left[ \exp(\lambda^{1/3} t) + \exp(\lambda^{1/3} \omega t) + \exp(\lambda^{1/3} \omega^2 t) \right], \text{ and so} \]
  \[ EA_t = \frac{\lambda^{-2/3}}{3} \left[ \exp(\lambda^{1/3} t) + \omega \exp(\lambda^{1/3} \omega t) + \omega^2 \exp(\lambda^{1/3} \omega^2 t) \right], \]

- $(X_t, L_t, A_t)$ is a Markov process.
- If $\mathcal{F}_s = \sigma\{X_r, L_r, A_r : r \leq s\}$, then
  \[
  \frac{d}{dt} E \left[ \begin{array}{c|c}
  X_t \\
  L_t \\
  A_t \\
  \end{array} \bigg| \mathcal{F}_s \right]_{t=s} = Q \left[ \begin{array}{c}
  X_s \\
  L_s \\
  A_s \\
  \end{array} \right], \text{ where } Q = \left( \begin{array}{ccc}
  0 & 0 & \lambda \\
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  \end{array} \right).
  \]

- The left eigenvalues of $Q$ are $\eta = \lambda^{1/3}, \lambda^{1/3} \omega, \lambda^{1/3} \omega^2$ with eigenvector $(1, \eta, \eta^2)$.
- From Dynkin's formula, $e^{-\eta t}(X_t + \eta L_t + \eta^2 A_t)$ is a (complex) martingale.
The random variable $M$

**Theorem**

$$M_t := \exp(-\lambda^{1/3} t)(X_t + \frac{\lambda^{1/3}}{3} L_t + \frac{\lambda^{2/3}}{3} A_t)$$ is a positive $L^2$-martingale, and so $M_t \to M$ a.s. and in $L^2$. $M$ does not depend on $\lambda$ and $P(M > 0) = 1$. $X_t/EX_t, L_t/EL_t, A_t/EA_t \to M$ a.s. and in $L^2$. 
Hitting time

- \( \tau(\epsilon) = \inf\{ t : C_t \geq \epsilon N^2 \} \). We compare it with
  \[ \sigma(\epsilon) := \inf\{ t : A_t \geq \epsilon N^2 \}. \]

- \( EA_t \sim a(t) = (1/3)N^{2\alpha/3} \exp(N^{-\alpha/3}t) \), and let
  \[ S(\epsilon) := N^{\alpha/3}[(2 - 2\alpha/3) \log N + \log(3\epsilon)] \] so that \( a(S(\epsilon)) = \epsilon N^2 \).

- Using the \( L^2 \) convergence we have nice estimates for
  \( P(\sup_{t \geq u} |A_t/a(t) - M| > \gamma) \) which in turn gives:

**Lemma**

*If \( 0 < \epsilon < 1 \), then as \( N \to \infty \),

\[ N^{-\alpha/3}(\sigma(\epsilon) - S(\epsilon)) \xrightarrow{P} - \log(M). \]

The coupling between \( C_t \) and \( A_t \) implies \( \tau(\epsilon) \geq \sigma(\epsilon) \).
Upper bound for $\tau(\varepsilon)$

To have an upper bound for $\tau(\varepsilon)$ we need to bound the difference $A_t - C_t$. Note that

- $x \in C_t$ if and only if a center is born in the space-time cone

$$K_{x,t} := \left\{ (y, s) \in \Gamma(N) \times [0, t] : |y - x| \leq (t - s)/\sqrt{2\pi} \right\}.$$

So

$$P(x \notin C_t | s_0, s_1, s_2, \ldots) = \prod_{i:s_i \leq t} \left[ 1 - \frac{(t-s_i)^2}{2N^2} \right] \leq \exp \left[ - \sum_{i:s_i \leq t} \frac{(t-s_i)^2}{2N^2} \right],$$

which gives a lower bound

$$EC_t \geq N^2 E \left[ 1 - \exp \left( - \int_0^t \frac{(t-s)^2}{2N^2} \, d\tilde{X}_s \right) \right].$$
Upper bound for $\tau(\epsilon)$ (continued)

- Using the inequality $1 - e^{-x} \geq x - x^2/2$ for $x \geq 0$,

  \[
  EC_t \geq \frac{t^2}{2} + \int_0^t \frac{(t - s)^2}{2} \, d\tilde{X}_s - \frac{1}{2N^2} E \left[ \int_0^t \frac{(t - s)^2}{2} \, dX_s \right]^2
  \]

  \[
  \geq \frac{t^2}{2} + \int_0^t \frac{(t - s)^2}{2} \lambda EC_s \, ds - \frac{EA_t^2}{2N^2}.
  \]

- Recall that by the definition of $A_t$,

  \[
  EA_t = \frac{t^2}{2} + \int_0^t \frac{(t - s)^2}{2} \lambda EA_s \, ds.
  \]

- So if $u(t) = EA_t - EC_t$, then

  \[
  u(t) \leq \frac{EA_t^2}{2N^2} + \int_0^t \frac{(t - s)^2}{2} \lambda u(s) \, ds = \frac{EA_t^2}{2N^2} + \int_0^t u(t - r) \lambda \frac{r^2}{2} \, dr,
  \]
Upper bound for $\tau(\epsilon)$ (continued)

- From the last argument

$$EC_t \geq EA_t - \frac{11a^2(t)}{N^2}.$$ 

- Using Markov inequality we can bound $A_t - C_t$, and have

**Lemma**

For any $\gamma > 0$,

$$\limsup_{N \to \infty} P[\tau(\epsilon) > \sigma((1 + \gamma)\epsilon)] \leq P \left( M \leq (1 + \gamma)\epsilon^{1/3} \right) + 11\frac{\epsilon^{1/3}}{\gamma}.$$ 

**Remark:** So $\tau(\epsilon) \sim (2 - 2\alpha/3)N^{\alpha/3} \log N.$
How does $C_t/N^2$ grow?

We choose

- $R := N^{\alpha/3}[(2 - 2\alpha/3) \log N - \log(M)]$ and $\psi(t) := R + N^{\alpha/3}t$ so that

$$A_{\psi(t)}/N^2 \xrightarrow{P} e^t/3, -\infty < t < \infty.$$ 

In particular for $W = \psi(\log(3\epsilon))$, $N^{-2}A_W \xrightarrow{P} \epsilon$.

- If $\epsilon$ is small, then using previous bound on $A_t - C_t$, $C_W/A_W$ is close to 1.
- Thus $C_W$ is close to $\epsilon N^2$ with high probability.

To study the growth of $C_t$ after time $W$,

- call the centers present at time $W$ ‘generation 0 centers’.
- For $k \geq 1$, generation $k$ centers are those which are born from area covered by generation $(k - 1)$ centers.
Estimates for area covered by generation 0 centers

**Def:** For \( k \geq 0 \) let \( C^k_{W,\psi(t)} \) (resp \( A^k_{W,\psi(t)} \)) be the area covered in \( C_t \) (resp \( A_t \)) by centers of generations \( j \in \{0, 1, \ldots, k\} \). Then

\[
A^0_{W,\psi(t)} = \int_0^W \frac{(\psi(t) - r)^2}{2} \, dX_r
\]

\[
= \frac{(\psi(t) - W)^2}{2}X_W + (\psi(t) - W)L_W + A_W
\]

\[
= \frac{(t - \log(3\epsilon))^2}{2}N^{2\alpha/3}X_W + (t - \log(3\epsilon))L_W + A_W.
\]

Recall that \( N^{-2}A_W \overset{P}{\to} \epsilon \). In the same spirit,

\( N^{-(2-\alpha/3)}L_W, N^{-(2-2\alpha/3)}X_W \overset{P}{\to} \epsilon \).

So if \( g_0(t) := \epsilon[1 + (t - \log(3\epsilon)) + (t - \log(3\epsilon))^2/2] \),

\[
\lim_{N \to \infty} P \left( \sup_{s \in [\log(3\epsilon), t]} \left| N^{-2}A^0_{W,\psi(s)} - g_0(s) \right| > \eta \right) = 0 \quad \text{for any } \eta > 0.
\]
Estimates for area covered by generation 0 centers

For a lower bound on $C_{W,\psi(t)}^0$, note that $x \in C_{s,t}^0$ if and only if a center is born in the space-time cone

$$K_{x,s,t} := \left\{(y, r) \in \Gamma(N) \times [0, s] : |y - x| \leq (t - r)/\sqrt{2\pi}\right\}.$$

Applying the inequality $1 - e^{-x} \geq x - x^2/2$ and doing some algebra,

$$EC_{s,t}^0 \geq EA_{s,t}^0 - \frac{a^2(s)}{N^2} p((t - s)N^{-\alpha/3}),$$

where $p(\cdot)$ is a polynomial.

This leads to the estimate

$$P\left(\inf_{s \in I_{\epsilon,t}} N^{-2} \left(C_{W,\psi(s)}^0 - A_{W,\psi(s)}^0\right) < -\epsilon^{7/6}\right) \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

Since $C_{W,\psi(t)}^0 \leq A_{W,\psi(t)}^0$, if $\epsilon$ is small, with high probability $g_0(t)$ and $f_0(t) := g_0(t) - \epsilon^{7/6}$ provide upper and lower bounds respectively for $N^{-2} C_{W,\psi(t)}^0$. 
Lower bound for $C^1_{W,\psi(t)}$

A point $x \notin C^1_{W,\psi(t)}$, if $x \notin C^0_{W,\psi(t)}$ and no generation 1 center is born in

$$K^\epsilon_{x,t} \equiv \left\{ (y, s) \in \Gamma(N) \times [W, \psi(t)] : |y - x| \leq (\psi(t) - s) / \sqrt{2\pi} \right\}.$$ 

To get a lower bound, we compare it with a process $B^1_{\psi(t)}$, where $B^0_t$ has area $N^2 f_0(t)$ and generation 1 centers are born as a Poisson process with intensity $N^{2-\alpha} f_0(\cdot)$.

Using a PP computation and second moment argument, if

$$1 - f_1(t) = (1 - f_0(t)) \exp \left( - \int_{\log(3\epsilon)}^t \frac{(t - s)^2}{2} f_0(s) \, ds \right),$$

the $N^{-2} |B^1_{\psi(t)}|$ is close to $f_1(t)$. This gives the estimate

$$\limsup_{N \to \infty} P \left[ \inf_{s \in I_{\epsilon,t}} (N^{-2} C^1_{W,\psi(s)} - f_1(s)) < -\delta \right] \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$

for any $\delta > 0$ and small $\epsilon$. 
Lower bound for $C_\psi(t)$

The last argument can be iterated. Let

$$1 - f_{k+1}(t) = (1 - f_k(t)) \exp \left( - \int_0^t \frac{(t - s)^2}{2 \log(3\epsilon)} (f_k(s) - f_{k-1}(s)) \, ds \right)$$

$$\cdots = (1 - f_0(t)) \exp \left( - \int_0^t \frac{(t - s)^2}{2 \log(3\epsilon)} f_k(s) \, ds \right).$$

$f_k \uparrow f_\epsilon$ satisfying

$$f_\epsilon(t) = 1 - (1 - f_0(t)) \exp \left( - \int_0^t \frac{(t - s)^2}{2 \log(3\epsilon)} f_\epsilon(s) \, ds \right)$$

with $f_\epsilon(0) = \epsilon - \epsilon^{7/6}$ and $|f_\epsilon(t) - f_k(t)| \leq \frac{(t - \log(3\epsilon))^{3k}}{(3k)!}$. Choosing $k$ large enough and noting that $C_\psi(t) \geq C^k_{\mathcal{W},\psi}(t)$, if $\epsilon$ is small and $\delta > 0$,

$$\limsup_{N \to \infty} P \left[ \inf_{s \in I_{\epsilon,t}} (N^{-2} C_\psi(s) - f_\epsilon(s)) < -\delta \right] \leq P(M < \epsilon^{1/3}) + \epsilon^{1/12}.$$
Recall that $g_0(\cdot) = \epsilon [1 + (\cdot - \log(3\epsilon)) + (\cdot - \log(3\epsilon))^2 / 2]$ is an upper bound of $C_{W,\psi}^0(t)$. Using similar argument, $g_k(\cdot)$ satisfying

$$1 - g_{k+1}(t) = (1 - g_k(t)) \exp \left( - \int_{\log(3\epsilon)}^{t} \frac{(t - s)^2}{2} (g_k(s) - g_{k-1}(s)) \, ds \right)$$

$$\cdots = (1 - g_0(t)) \exp \left( - \int_{\log(3\epsilon)}^{t} \frac{(t - s)^2}{2} g_k(s) \, ds \right)$$

provides an upper bound for $C_{W,\psi}(\cdot)$. 

$g_k \uparrow g_\epsilon$ satisfying

$$g_\epsilon(t) = 1 - (1 - g_0(t)) \exp \left( - \int_{\log(3\epsilon)}^{t} \frac{(t - s)^2}{2} g_\epsilon(s) \, ds \right)$$

with $g_\epsilon(0) = \epsilon$ and $|g_\epsilon(t) - g_k(t)| \leq \frac{(t - \log(3\epsilon))^{3k}}{(3k)!}$. 
Upper bound for $C_{\psi}(t)$ (continued)

$C^k_{W,\psi}(t) \uparrow C_{\psi}(t)$ uniformly in $k$, as

- $C_{\psi}(t) - C^k_{W,\psi}(t) \leq A_{\psi}(t) - A^k_{W,\psi}(t)$
- $A_{s+t} - A^k_{s,s+t}$ is increasing in $s$ and decreasing in $k$.

So choosing large $k$, if $\epsilon$ is small and $\delta > 0$,

$$\limsup_{N \to \infty} P \left[ \sup_{s \in I_{\epsilon,t}} \left( N^{-2} C_{\psi}(s) - g_\epsilon(s) \right) > \delta \right] \leq P(M < \epsilon^{1/3}) + \epsilon^{2/3}.$$
Limiting behavior of $C_{\psi}(t)$

$g_\epsilon(t)$ and $f_\epsilon(t)$ provide upper and lower bounds for $C_{\psi}(t)$. In the limit as $\epsilon \to 0$ both the bounds converge to the same thing. Let $h_\epsilon(t) = e^t/3$ for $t < \log(3\epsilon)$.

$$h_\epsilon(t) = 1 - \exp \left( - \int_{-\infty}^{\log(3\epsilon)} \frac{(t - s)^2 e^s}{2} ds - \int_{\log(3\epsilon)}^{t} \frac{(t - s)^2}{2} h_\epsilon(s) ds \right)$$

for $t \geq \log(3\epsilon)$. Then

$$\sup_{s \in I_{\epsilon, t}} |f_\epsilon(s) - h_\epsilon(s)|, \sup_{s \in I_{\epsilon, t}} |g_\epsilon(s) - h_\epsilon(s)| \to 0, \quad h_\epsilon(t) \to h(t)$$

satisfying (a) $\lim_{t \to -\infty} h(t) = 0$ (b) $\lim_{t \to \infty} h(t) = 1$ (c) $h$ is increasing with $0 < h(t) < 1$ and

$$(d) \quad h(t) = 1 - \exp \left( - \int_{-\infty}^{t} \frac{(t - s)^2}{2} h(s) ds \right).$$
Limiting behavior of $C_{\psi}(t)$

**Theorem (C. and Durrett; to appear in AoAP)**

For any $t < \infty$ and $\delta > 0$,

$$
\lim_{N \to \infty} P \left( \sup_{s \leq t} \left| N^{-2} C_{\psi}(s) - h(s) \right| \leq \delta \right) = 1.
$$

**Remarks:**

- The displacement of $\tau(\epsilon)$ from $(2 - 2\alpha/3)N^{\alpha/3} \log N$ on the scale $N^{\alpha/3}$ is dictated by the random variable $M$ that gives the rate of growth of the balloon branching process.
- Once $C_t$ reaches $\epsilon N^2$, the growth is deterministic.
The cover time $T_N$

- $h(t)$ never reaches 1.
- Since $N^{-2} C_{\psi(s)} \sim h(s)$, the number of centers in $C_{\psi(0)}$ dominates a Poisson random variable with mean
  
  $$\lambda(\delta) N^{2-2\alpha/3},$$
  where
  $$\lambda(\delta) = \int_{-\infty}^{0} (h(s) - \delta)^+ \, ds,$$

  which are uniformly distributed in the torus.
- If $\delta > 0$ is small, then $\lambda(\delta) > 0$.
- Divide the torus into smaller squares with side $\kappa N^{\alpha/3} \sqrt{\log N}$.
- With high probability each of the small squares owns at least one center at time $\psi(0)$.
- This makes $T_N \leq \psi(0) + O(N^{\alpha/3} \sqrt{\log N})$, and so
  $$T_N / N^{\alpha/3} \log N \to 2 - 2\alpha/3.$$
Future direction

What happens when percolation rate depends on distance?
Thank You